

Spectral function of a scalar boson coupled to fermions

Francesco Giacosa⁽¹⁾ and Giuseppe Pagliara⁽²⁾

⁽¹⁾*Institut für Theoretische Physik*

Johann Wolfgang Goethe - Universität, Max von Laue-Str. 1

D-60438 Frankfurt, Germany

⁽²⁾*Dip. di Fisica dell'Università di Ferrara and*

INFN Sez. di Ferrara, Via Saragat 1, I-44100 Ferrara, Italy

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Abstract

We present the calculation of the spectral function of an unstable scalar boson coupled to fermions as resulting from the resummation of the one loop diagrams in the scalar particle self energy. We work with a large but finite high-energy cutoff: in this way, the spectral function of the scalar field is always correctly normalized to unity, independently on the value of the cutoff. We show that this high energy cutoff affects the Breit-Wigner width of the unstable particle: the larger the cutoff, the smaller is the width at fixed coupling. Thus, the existence of a high energy cutoff (alias minimal length), and for instance the possible opening of new degrees of freedom beyond that energy scale, could then be in principle proven by measuring, at lower energy scales, the line shape of the unstable scalar state. Finally, we briefly discuss the case of the Higgs boson and we show that in that case the influence of the cutoff is very small and non detectable. However, it is conceivable that in extensions of the Standard model, broader resonances exist: then, the described effect could be measured.

1 Introduction

The aim of this work is to study the spectral function of a scalar field, denoted as H , coupled via a simple renormalizable Yukawa-type interaction to a fermion field ψ :

$$\mathcal{L}_{int} = gH\bar{\psi}\psi . \quad (1)$$

We assume that the scalar boson H is heavy enough for the decay process $H \rightarrow \bar{\psi}\psi$ to take place. Thus, H is unstable and has not a definite mass: a spectral function $d_H(x)$ can be obtained as the imaginary part of the propagator of H . Intuitively, the quantity $d_H(x)dx$ represents the ‘mass distribution’, that is the probability that the unstable state H has a mass between x and $x+dx$ [1, 2, 3]. While the Breit-Wigner function represents often a good approximation for $d_H(x)$, deviations become evident when a more advanced treatment of the problem is undertaken. A natural condition which must be fulfilled is the normalization equation:

$$\int_0^\infty d_H(x)dx = 1 , \quad (2)$$

which assures the normalization of the probability associated to the mass distribution, i.e. the normalization of the initial unstable state H . As we shall see, interesting effects connected to Eq. (2) emerge when studying this system in detail.

The determination of the propagator of H is a necessary step to obtain its spectral function, which is proportional to the imaginary part of the propagator. We consider a fermionic loop which dresses the bare propagator of H and we perform a resummation of this loop contribution. Simple power counting shows that the fermionic loop is divergent. Thus, one has to cure the divergences according to a certain regularization. In this work we shall use the old-fashioned cutoff regularization¹: we thus introduce a finite, albeit large, high energy scale Λ . Namely, we argue that a finite cutoff is better suited to describe a physical situation, in which high energy contributions are effectively suppressed when the energy of the particles circulating in the loop is high enough. Although its precise value, and also the way the high momenta are suppressed are unknown (a typical choice consists in taking Λ equal to the Planck mass), the finiteness of the cutoff assures that the condition (2) is always fulfilled. In turn, a logarithmic dependence on the cutoff cannot be eliminated (by using relations between bare and dressed parameters): namely, we find that the form of $d_H(x)$ is (weakly) influenced by the precise value of the cutoff. Before discussing this main property of our results in detail we briefly recall the ideas behind regularization and renormalization and justify the use of a finite cutoff.

The appearance of divergences in Quantum Field Theory (QFT) plagued its first stages, up to the development of a successful renormalization program [4, 5, 6, 7]. The first step of the renormalization is the regularization procedure, in which the divergent integrals appearing at high orders in perturbation theory are made finite according to a certain prescription: in the already mentioned cutoff regularization the momenta of the virtual particles are ‘cut’ for high values beyond a certain ultraviolet (UV) energy scale (the cutoff) Λ ; in the Pauli-Villars approach particles with a large mass Λ_{PV} (which plays the role of the high energy scale in this scheme) are formally introduced in such a way that the ultraviolet contributions cancel; in the dimensional regularization the integrals are evaluated in $4 - \epsilon$ dimensions and the divergences appear as ϵ^{-1} contributions. The use of a certain regularization scheme depends on the problem under study. In fact, a regularization can ‘destroy’ some original symmetries of the Lagrangian, and therefore care is needed. For instance, a cutoff Λ violates gauge invariance (its restoration is indeed possible, but lengthy [8, 9]), while the Pauli-Villars and dimensional regularizations preserve it and are therefore usually preferred in explicit calculations in the framework of gauge theories (though the Pauli-Villars does not preserve gauge invariance in non-abelian gauge theories).

Once a QFT Lagrangian has been regularized, one can reabsorb the divergences into the bare parameters of the theory (masses and couplings) plus the wave-function renormalizations (these steps can be also done by introducing proper counterterms, which order by order assure that the divergences disappear). At this point the high energy scale has disappeared from the QFT and can be formally set to infinity. Each quantity is perfectly finite and independent on Λ (or on Λ_{PV} and ϵ). It is well known that, only for a small subset of QFTs, the renormalizable theories, this procedure is possible and no divergence (i.e., explicit dependence on the high energy scale) emerges at higher orders. Indeed, the Lagrangian of the Standard Model (SM) contains only renormalizable interactions (see e.g. Ref. [10] and refs. therein).

Non-renormalizable QFTs were regarded in the past as substantially ill-defined because the high-energy scale does not decouple. Formally, one could introduce at each order new counterterms, but the price is the need to introduce new coupling constants at each order.

¹Of course, more sophisticated and effective regularization procedures exist (as described later on and in the Appendix) and are commonly used for calculations. However, the main aim of this work is conceptual and we thus wish to explicitly keep track of a finite high-energy scale.

However, it is interesting to stress that the point of view toward non-renormalizable theories changed in the last decades. Especially in the framework of Quantum Chromodynamics (QCD), the development of a variety of QFTs which are not renormalizable was put forward in order to model nonperturbative QCD phenomena: (i) chiral perturbation theory is constructed as a theory of the lightest hadronic states (the pions in its simplest form) [11, 12]; the Lagrangian is organized order-by-order with increasing number of derivatives, which in turn implies an increasing number of pion momenta in the corresponding Feynman diagrams. The Lagrangian of chiral perturbation theory is non-renormalizable, but a successful renormalization program can be carried out order by order. (ii) The Nambu Jona-Lasinio model is a model of quarks with a quartic, Fermi-like (non-renormalizable) interaction. A finite QCD-driven cutoff of about 600 MeV is introduced to correctly describe the vacuum's phenomenology, see for instance Ref. [13]. (iii) Although the original σ model was renormalizable [14], modern versions of it are not [15].

More in general, nowadays also the SM itself is regarded as an effective model of a yet-unknown theory which represents its ultraviolet completion. It is indeed known that at energy larger than the Planck mass gravity effects are non-negligible. Although a quantum theory of gravity is still not available, we can conclude that the cutoff of the Standard Model Λ_{SM} should be smaller than the Planck mass, $\Lambda_{SM} \lesssim M_{Planck}$. But this is an upper limit: Λ_{SM} could be much smaller than that, up to the order of 1 TeV = 10^3 GeV. It is then plausible to conclude that Λ_{SM} lies in the (quite broad) range $(10^3, 10^{19})$ GeV.

However, as long as the cutoff Λ (or, equivalently, Λ_{PV}) in a renormalizable theory is finite but much larger than other dimensionful parameters of the theory, then the results should depend on it at most as $1/\Lambda$, $1/\Lambda^2$, ... and are therefore very difficult to be seen in low-energy processes. (Moreover, such contributions obviously vanish when the formal limit $\Lambda \rightarrow \infty$ is taken.) Thus, the cutoff is a physical energy scale, which however does not affect the low-energy behavior of the theory. This point of view is very well described in the QFT book by Zee [7], where it is stressed that the regularization is not only a mathematical intermediate step but corresponds in some sense to a physical situation: *“I emphasize that Λ should be thought of as physical, parametrizing our threshold of ignorance, and not as a mathematical construct. Indeed, physically sensible quantum field theories should all come with an implicit Λ . If anyone tries to sell you a field theory claiming that it holds up to arbitrarily high energies, you should check to see if he sold used cars for a living (pages 146-147 in Ref. [7]).”*

Having clarified and motivated why we insist on working with a finite cutoff Λ , we come back to the purpose of the present work: namely, we aim to investigate which role plays the cutoff Λ on the spectral function $d_H(x)$ of the unstable scalar state H . At a first sight, this seems a ill-posed question, cause the cutoff Λ should not affect, for all the reasons described above, a physical quantity such as the spectral function. (In fact, $d_H(x)$ can be related -for instance- to fermionic pair-production process, whose cross section is described in Sec. 2.4.) Quite surprisingly, we find that this is not the case, and that $d_H(x)$ has a logarithmic dependence on the cutoff: the finite value of Λ influences the width of the peak of the function $d_H(x)$. It is then conceivable that one may pin down the value of the high energy scale Λ by studying the spectral function of the low-energy resonance H . Note, this peculiar dependence on the cutoff does not take place in superrenormalizable theories, in which the value of Λ does not affect the form of the spectral function if it is large enough [3]. We shall also show that the limit $x \rightarrow \infty$ and $\Lambda \rightarrow \infty$ do not commute. When $\Lambda \rightarrow \infty$ is taken first, and consequently the standard renormalization procedure is applied, no dependence on the cutoff is left, but a series of inconsistencies emerges: the spectral function is not localized in the vicinity of the peak, neither for small values of the coupling constant. This result represents a further hint toward the existence of a finite cutoff.

An immediate application of our formulae can be done in the case of the Higgs boson, which has a coupling of the type of Eq. (1) to fermions. However, for the determined Higgs mass of 125 GeV [16] it turns out that the Higgs spectral function is very narrow and thus very well approximated by a simple Breit-Wigner form. The dependence on the cutoff, although present in principle, cannot be seen in practice, because its influence on the form of $d_H(x)$ is vanishingly small. It is interesting to notice that if the Higgs mass would have been higher (500 GeV or more), than the resonance would have been broad: deviations from the Breit-Wigner form and the influence of the -albeit large but finite- cutoff would have been more pronounced. On the other hand it is conceivable that other (pseudo)scalar resonances beyond the minimal SM exist, which are broad and thus could show a direct dependence of the cutoff in their spectral function.

The paper is organized as follows: in Sec. 2 we present the model and the calculation of the self-energy and spectral function. In Sec. 3 we show the numerical results for some interesting cases and finally in Sec. 4 we draw our conclusions and possible future developments. A rich appendix is also included in which we discuss different technicalities and subtle points for the interested reader.

2 The model and its implications

2.1 The Lagrangian

We study the following renormalizable Lagrangian in which the scalar particle H (with bare mass $M_{0,H}$) is coupled to the fermion field ψ (with mass m_f):

$$\mathcal{L} = \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}M_{0,H}^2 H^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m_f)\psi + gH\bar{\psi}\psi, \quad (3)$$

where g is the dimensionless coupling constant. Thus, the Lagrangian describes a simple Yukawa interaction of a massive fermion with a massive scalar boson.

2.2 Decay width

As a first step we evaluate the tree-level decay width for the process $H \rightarrow \bar{\psi}\psi$. For future purposes we evaluate it for the arbitrary mass x of the particle H :

$$\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x) = \frac{\sqrt{\frac{x^2}{4} - m_f^2}}{8\pi x^2} (4m_f^2) \sum_{\alpha,\beta} |\mathcal{M}^{\alpha\beta}|^2 \theta(x - 2m_f) \quad (4)$$

where the amplitude reads:

$$-i\mathcal{M}^{\alpha\beta} = -ig\bar{u}^{(\alpha)}(\vec{k}_1)v^{(\beta)}(\vec{k}_2) \quad (5)$$

Following the usual steps (details in Appendix A.1) the tree-level decay width $\Gamma_{H \rightarrow \bar{\psi}\psi}(x)$ as function of the (running) mass x reads:

$$\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x) = \frac{\left(\frac{x^2}{4} - m_f^2\right)^{3/2}}{\pi x^2} g^2 \theta(x - 2m_f). \quad (6)$$

Naively, the on-shell tree-level decay width is evaluated by setting $x = M_{0,H}$. However, care is needed because it is a well known fact that the mass of the H field is modified by loop corrections. In particular, we will see in the subsections 2.3 and 2.4 that:

$$M_{0,H} \xrightarrow{\text{loops}} M_H < M_{0,H}, \quad (7)$$

i.e. the loops reduce the mass. The numerical value of the tree-level decay width is obtained by evaluation the tree-level decay function at the dressed mass M_H (and not at the bare mass $M_{0,H}$): $\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x = M_H)$.

The spectral function, to be studied in details later, can be approximated by the following schematic behavior:

$$d_H^{\text{appr}}(x) \simeq \frac{2x}{\pi} \frac{x \Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x)}{(x^2 - M_H^2)^2 + x^2 \left(\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x) \right)^2}, \quad (8)$$

where the real part and cutoff-effects in the imaginary part of the loop have been neglected. For large x , the approximate asymptotic behavior $d_H^{\text{appr}}(x) \sim 1/x$ holds because the decay function $\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x)$ scales as $\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x) \sim x$. Such a spectral function is clearly non normalized. We shall elaborate on this issue more in detail in the following, where we will show that the presence of a cutoff (no matter how large) assures that the correct normalization $\int_0^\infty d_H(x) dx = 1$ is obtained.

2.3 The fermionic loop

The scalar state H is dressed by fermion loops. The contribution of one fermion loop is easily evaluated by using the Feynman rules:

$$\Sigma(p) = i \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [\Delta_f(q + p/2) \Delta_f(q - p/2)] , \quad (9)$$

where the fermion propagator reads

$$\Delta_f(q) = \frac{1}{\gamma^\mu q_\mu - m_f + i\varepsilon} . \quad (10)$$

The integral in Eq. (9) is quadratically divergent. It must be therefore regularized; for the reasons described in the Introduction we use here a regularization function $\phi_\Lambda(p, q)$, which depends on the cutoff Λ :

$$\Sigma(p) = i \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [\Delta_f(q + p/2) \Delta_f(q - p/2)] \phi_\Lambda^2(p, q) . \quad (11)$$

Upon one-loop resummation the propagator of H takes the form

$$\Delta_H(p) = \frac{1}{p^2 - M_{0,H}^2 + g^2 \Sigma(p) + i\varepsilon} , \quad (12)$$

where the loop $\Sigma(p)$ can be rewritten in the following way:

$$\Sigma(p) = i \int \frac{d^4 q}{(2\pi)^4} \frac{\text{Tr} \left[[\gamma^\mu (q_\mu + \frac{1}{2} p_\mu) + m_f] [\gamma^\nu (q_\nu - \frac{1}{2} p_\nu) + m_f] \right]}{[(q + p/2)^2 - m_f^2 + i\varepsilon] [(q - p/2)^2 - m_f^2 + i\varepsilon]} \phi_\Lambda^2(p, q) . \quad (13)$$

For what concerns $\phi_\Lambda(p, q)$ we make here the following assumption:

$$\phi_\Lambda(p, q) = f_\Lambda \left(\frac{q^2 p^2 - (q \cdot p)^2}{p^2} \right) . \quad (14)$$

Notice that the function is expressed in terms of scalar products of four-vectors and it is thus manifestly covariant. On a practical level we use the following form for $f_\Lambda(\eta)$:

$$f_\Lambda(\eta) = \theta(\eta + \Lambda^2) , \quad (15)$$

where Λ is a cutoff, see Appendix A.2.1 for more technical details. The choice in Eq. (15) is simple and allows for an analytic presentation of many formulae. However, one could have used smooth and more complicated cutoff functions, see for instance Refs. [3] and refs. therein. Only small numerical changes would be found but no conceptual changes would follow.

The trace in the integral (13) reads:

$$\begin{aligned} & \text{Tr} \left[\left[\gamma^\mu \left(q_\mu + \frac{1}{2} p_\mu \right) + m_f \right] \left[\gamma^\nu \left(q_\nu - \frac{1}{2} p_\nu \right) + m_f \right] \right] \\ &= 4 \left(q + \frac{p}{2} \right) \cdot \left(q - \frac{p}{2} \right) + 4m_f^2 = 4 \left(q^2 - \frac{p^2}{4} + m_f^2 \right) . \end{aligned} \quad (16)$$

Then, working in the reference frame of the particle H , for which $p = (x, 0) \rightarrow p^2 = x^2$, and performing the integral over q^0 by utilizing the residues calculus, one finds:

$$\Sigma(x) = \int \frac{d^3 q}{(2\pi)^3} \frac{4f_\Lambda^2(-\mathbf{q}^2)}{2x\sqrt{\mathbf{q}^2 + m_f^2}} \frac{4x\mathbf{q}^2}{[4(\mathbf{q}^2 + m_f^2) - x^2 + i\varepsilon]} , \quad (17)$$

where we have taken into account that, in the rest frame of H , one has $\phi_\Lambda(p, q) = f_\Lambda(-\mathbf{q}^2)$ (i.e., no explicit dependence on q^0 is present). Introducing the variable w defined as $w^2 = \mathbf{q}^2$ we rewrite the loop as:

$$\Sigma(x) = \frac{1}{2\pi^2} \int_0^\infty dw \frac{8f_\Lambda^2(-w^2)}{\sqrt{w^2 + m_f^2}} \frac{w^4}{(4(w^2 + m_f^2) - x^2 + i\varepsilon)} . \quad (18)$$

The quadratic divergence of the loop is again clear. The validity of the optical theorem

$$x\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x) f_\Lambda \left(-\sqrt{\frac{x^2}{4} - m_f^2} \right) = g^2 \text{Im} \Sigma(x) \quad (19)$$

can be easily verified from Eq. (18) by an explicit calculation of the imaginary part.

Strictly speaking, the ‘correct’ tree-level decay width, including the effect of the cutoff function, is given by

$$\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1,correct}}(x) = \Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-1}}(x) f_\Lambda \left(-\sqrt{\frac{x^2}{4} - m_f^2} \right) \quad (20)$$

where the vertex-function directly enters into the expression. This result can be achieved by using a nonlocal Lagrangian and its corresponding Feynman rules, see Appendix A.2.1 and Refs. [3, 9, 17].

In this work we make the choice in Eq. (15), for which an analytic form of the loop is obtained as:

$$\begin{aligned} \Sigma(x) = & \frac{-1}{4\pi^2 x} \left\{ (x^2 - 4m_f^2)^{3/2} \text{arctanh} \left[\frac{\Lambda x}{\sqrt{(\Lambda^2 + m_f^2)(x^2 - 4m_f^2)}} \right] \right. \\ & \left. + x \left[-2\Lambda \sqrt{\Lambda^2 + m_f^2} + (6m_f^2 - x^2) \ln \left(\frac{\Lambda + \sqrt{\Lambda^2 + m_f^2}}{m_f} \right) \right] \right\} . \end{aligned} \quad (21)$$

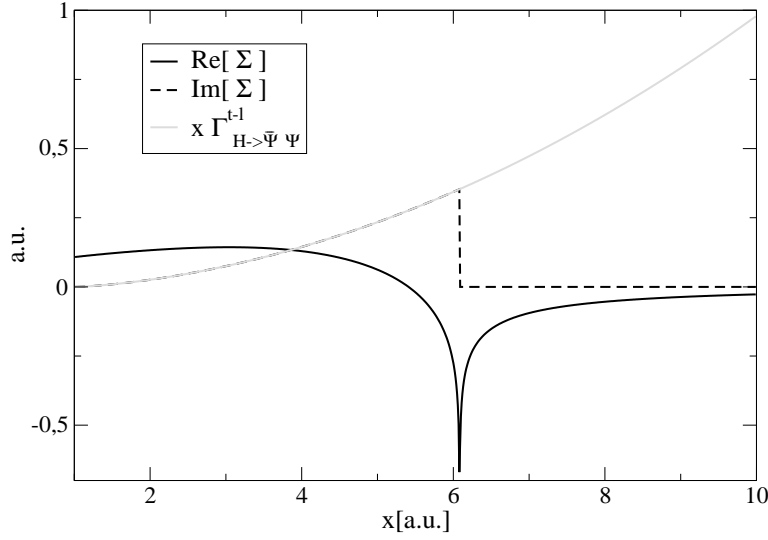


Figure 1: Real and imaginary part of the self energy. The quantity $x\Gamma_{H\rightarrow\bar{\Psi}\Psi}^{t-l}(x)$ is also shown to confirm the validity of the optical theorem. Because of the cutoff, introduced for the sake of having a correct normalization of the spectral function, the imaginary part of the self energy vanishes at high energies. Here $M_H = 1.5$ and $\Lambda = 3$, where the unit $2m = 1$ has been used.

We shall use the previous form for analytic and numerical calculations. Note that, as long as x fulfills the inequality

$$-\sqrt{\frac{x^2}{4} - m_f^2} + \Lambda > 0 \rightarrow x < 2\sqrt{\Lambda^2 + m_f^2} \lesssim 2\Lambda \quad (22)$$

one has for the adopted choice of the vertex function that $\Gamma_{H\rightarrow\bar{\psi}\psi}^{t-l,\text{correct}}(x) = \Gamma_{H\rightarrow\bar{\psi}\psi}^{t-l}(x)$. However, for $x > 2\sqrt{\Lambda^2 + m_f^2}$ the correct tree-level decay function vanishes. Thus, for large values of the cutoff the equality $\Gamma_{H\rightarrow\bar{\psi}\psi}^{t-l,\text{correct}}(x) = \Gamma_{H\rightarrow\bar{\psi}\psi}^{t-l}(x)$ holds for a very wide energy range. (Note, using a smooth cutoff function the strict equality $\Gamma_{H\rightarrow\bar{\psi}\psi}^{t-l,\text{correct}}(x) = \Gamma_{H\rightarrow\bar{\psi}\psi}^{t-l}(x)$ would hold only approximately in a wide energy region.)

In Fig. 1 we show the results for the real and imaginary part of the self energy for the following choice of the free parameters: $2m_f \equiv 1$ sets an arbitrary energy unit, $M_H = 1.5$ and $\Lambda = 3$. Notice that the imaginary part vanishes for values of the energy larger than $2\sqrt{\Lambda^2 + m_f^2}$ as explained before. The gray line represents the quantity $x\Gamma_{H\rightarrow\bar{\psi}\psi}^{t-l}(x)$ which, due to the optical theorem, is equal to the imaginary part of the self energy up to $2\sqrt{\Lambda^2 + m_f^2}$. The real part becomes also very small for $x \gtrsim 2\Lambda$: this is a crucial property to show the correct normalization, as it is presented in the next subsection and in the Appendix A.3.

A closer inspection of the loop expression (21) shows that constant terms can still be re-absorbed in the bare mass $M_{0,H}$ and have therefore no physical consequences; however, this does not hold for the term proportional to $x^2 \ln \Lambda$, which is responsible for a mixing of two, in principle well separated energy scales, i.e. x (which is the invariant mass in a scattering experiment, see the following discussion) and the cutoff Λ . Thus, in our scheme there is a logarithmic dependence of the cutoff in the loop formula and, consequently, on the form of the spectral function $d_H(x)$. This is indeed crucial for our results because this dependence on the high energy scale Λ does not decouple. Indeed, in Ref. [7] it was stated that the physical results (such as scattering lengths) do depend on the cutoff in a power-suppressed form $1/\Lambda$,

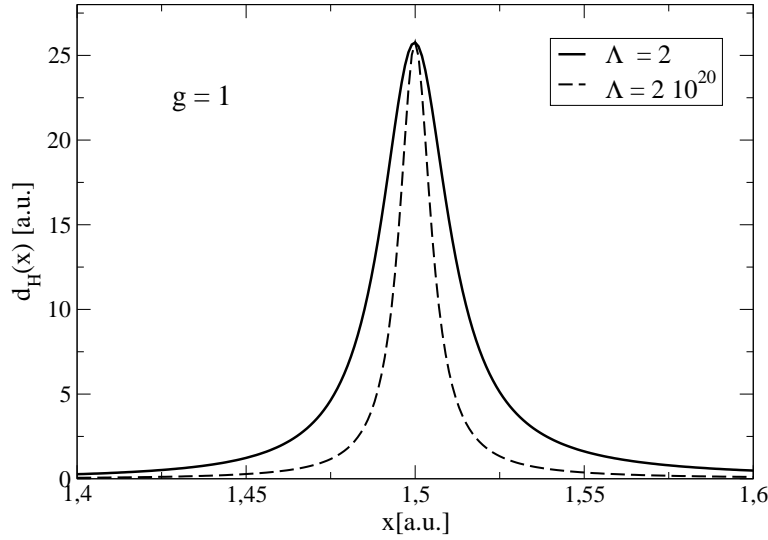


Figure 2: Spectral functions at fixed coupling $g = 1$ and for two very different choices of the cutoff value. Notice the narrowing of the spectral function as the cutoff is increased. M_H as in Fig. 1.

$1/\Lambda^2, \dots$. The new point here is that we find a dependence of the cutoff which is logarithmic and not power-like suppressed. Of course, a logarithmic dependence is weak, but can lead to interesting phenomena, as we shall see in Sec. 3.

2.4 The spectral function and its normalization

The spectral function (or mass distribution) of the scalar field H , denoted as $d_H(x)$, is defined as:

$$d_H(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{2x}{\pi} |\text{Im } \Delta_H(x)|. \quad (23)$$

Explicitly:

$$d_H(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{2x}{\pi} \frac{g^2 |\text{Im } \Sigma(x)|}{[x^2 - M_{0,H}^2 + g^2 \text{Re } \Sigma(x)]^2 + [g^2 \text{Im } \Sigma(x)]^2 + i\varepsilon}. \quad (24)$$

We define the nominal mass M_H of the resonance H as the zero of the real part of the propagator's denominator²:

$$x^2 - M_{0,H}^2 + g^2 \text{Re } \Sigma(x) = 0 \rightarrow x = M_H. \quad (25)$$

In general, the quantum loop generates a negative contribution, therefore $M_H < M_{0,H}$.

We assume here that $\text{Im } \Sigma(x = M_H) \neq 0$ (we are thus above threshold: $M_H > 2m$ and the decay channel $H \rightarrow \bar{\psi}\psi$ is open). Then, the limit can be easily performed:

$$d_H(x) = \frac{2x}{\pi} \frac{g^2 |\text{Im } \Sigma(x)|}{[x^2 - M_{0,H}^2 + g^2 \text{Re } \Sigma(x)]^2 + [g^2 \text{Im } \Sigma(x)]^2}. \quad (26)$$

In the limit $g \rightarrow 0$ the expected result $d_H(x) = \delta(x - M_{0,H})$ is obtained. One can compare Eq. (26) with the previously approximate version in Eq. (8): besides the real part, which is neglected in the approximate form, the two expressions coincide in virtue of the optical theorem for $x < 2\sqrt{\Lambda^2 + m_f^2}$.

²See Ref. [18] for a related study in which, instead of the nominal mass, the pole on the II-Riemann sheet is investigated.

We now turn to the normalization of $d_H(x)$, i.e. to the validity of the equation:

$$\int_0^\infty dx d_H(x) = 1 . \quad (27)$$

To this end we recall that the propagator $\Delta_H(x)$ can be expressed via the so-called Källén-Lehman representation [6]:

$$\Delta_H(x) = \int_0^\infty dy \frac{d_H(y)}{x^2 - y^2 + i\varepsilon} , \quad (28)$$

which intuitively corresponds to expressing the full propagator as the ‘sum’ of free propagators of the form $[x^2 - y^2 + i\varepsilon]^{-1}$, each of them weighted with the mass distribution $d_H(y)$. The physical interpretation of $d_H(y)dy$ as the probability that the unstable state H has a mass between y and $y + dy$ is evident. When considering the limit $x \rightarrow \infty$ the propagator can be approximated as

$$\Delta_H(x) \stackrel{x^2 \rightarrow \infty}{\simeq} \frac{1}{x^2} \quad (29)$$

provided that a finite (no matter how large) cutoff Λ is employed. In fact, in the case of a finite cutoff one has that $g^2 \text{Im } \Sigma(x^2) = x \Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-l,correct}} = 0$ for $x > 2\sqrt{\Lambda^2 + m_f^2}$, see the previous Section, and also the real part of $\Sigma(x)$ goes rapidly to zero for $x \rightarrow \infty$. When Eq. (29) holds (i.e. the cutoff is finite), Eq. (28) reduces to

$$\Delta_H(x \rightarrow \infty) = \frac{1}{x^2} = \int_0^\infty dy \frac{d_H(y)}{x^2} \rightarrow \int_0^\infty dy d_H(y) = 1 . \quad (30)$$

(See also the Appendix A.3 for a rigorous proof and for its extension to the case of a generic cutoff function, as long it vanishes sufficiently fast for large values of $x \gtrsim \Lambda$.)

The case of a large but finite cutoff corresponds to realistic cases. In fact, the cutoff within a ‘fundamental renormalizable theory’ signalizes the own breaking of the theory and its numerical value is typically much larger than the others energy scales of the theory (such as the Planck mass). Moreover, we have shown that the finite cutoff (independently on its value) assures that the mass distribution is already correctly normalized to unity. There is no need of a field strength renormalization in this framework; see next Section for numerical examples.

Some important points need to be discussed.

Standard renormalization treatment: It is possible, using the standard procedure, to remove each dependence on the cutoff. However, we shall show here and in the Appendix A.4 that inconsistencies arise. The first step consists in choosing a very large cutoff, $\Lambda \gg M_H$, which allows to simplify the formula (21) for $x \ll \Lambda$ as follows:

$$\tilde{\Sigma}(x) = -\frac{(x^2 - 4m_f^2)^{3/2}}{4\pi^2 x} \text{arctanh} \left(\frac{x}{\sqrt{x^2 - 4m_f^2}} \right) + \frac{\Lambda^2}{2\pi^2} + \frac{(x^2 - 6m_f^2)}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) \quad (31)$$

Taking the limit $\Lambda \rightarrow \infty$, the propagator corrections require a quadratically divergent mass renormalization to reabsorb the term $\sim \Lambda^2$ and a field strength renormalization to reabsorb the term $\sim x^2 \ln \Lambda$ [4]. This way can be easily followed in the case of a stable scalar particle, $M_H < 2m_f$: these two operations correspond to the conditions that the pole of the propagator occurs at M_H and that the residue at the pole is 1. When $M_H > 2m_f$ (i.e., H is unstable) this approach can be formally generalized, although it is not evident which constraint should be imposed to fix the renormalization constants. We shall discuss the possibilities in Sec. A.4 where we describe in detail the relevant procedure and formulae.

After the renormalization procedure, no dependence on the cutoff is present and the limit for large x of the loop function reads $\Sigma(x) \sim x^2 \ln x$ (and does not vanish as Eq. (21) does.) As a consequence, the propagator in this case has a different scaling than the one in Eq. (29): $\Delta_H(x) \stackrel{x^2 \rightarrow \infty}{\simeq} 1/(x^2 \ln x)$. Thus, the limits $x \rightarrow \infty$ and $\Lambda \rightarrow \infty$ do *not* commute: this is indeed the main origin of the (very) different results obtained in our framework and the ones of the standard renormalization. Moreover, due to the different scaling law, the correct normalization of the spectral function is not anymore guaranteed. Although the integral of the spectral function is still (slowly) convergent, one finds rather unphysical results: only a minimal part of the normalization of the spectral function is located in the vicinity of the peak, implying that the probability to excite the resonance at energies close to its nominal mass (corresponding to the position of the peak) is very small. Moreover, also the dependence of the normalization on the coupling constant g turns out to be unexpected: the smaller g , the smaller is the probability that the unstable state H has a mass close to the peak. We regard these properties as unphysical.

Other regularization procedures: For completeness we have performed in the Appendix, Secs. A.2.2 and A.2.3, the calculation of the loop in the Pauli-Villars and the dimensional regularization schemes. In both cases, as expected, the imaginary part coincides with that obtained in the cutoff scheme if it is sent to infinity, in agreement with the optical theorem. In the Pauli-Villars a very similar expression for $d_H(x)$ with a finite cutoff Λ is obtained in the vicinity of the peak (including a logarithmic term of the type $x^2 \ln \Lambda_{PV}$, as long as the corresponding cutoff Λ_{PV} is finite). However, the Pauli-Villars approach breaks unitarity already at the level of the Lagrangian and for $\Lambda_{PV} \rightarrow \infty$ the very same problems described above and in Appendix A.4 arise. In the dimensional regularization, instead of the term proportional to $x^2 \ln \Lambda$, a similar term proportional to x^2/ϵ is present: removing the latter is also completely equivalent to the case described above and in Appendix A.4.

Gedanken experiment: One may ask to which extent the spectral function $d_H(x)$ is a physical quantity. To show that $d_H(x)$ can be considered such, we present a ‘Gedanken experiment’, in which the introduced mass distribution $d_H(x)$ directly enters into the form of the total cross section. To this end, let us consider a (for simplicity massless) scalar field φ which is coupled to the scalar field H via the following interaction term:

$$\mathcal{L}_{int,\varphi} = \lambda H \varphi^2. \quad (32)$$

We assume here that the coupling constant λ is very small in such a way that the loop corrections arising from the loop of φ -fields are negligible when compared to the ψ -loop evaluated previously. Similarly, the decay width $\Gamma_{H \rightarrow \varphi\varphi}$ is assumed to be much smaller than $\Gamma_{H \rightarrow \bar{\psi}\psi}$ and can be then safely neglected. Writing $x = \sqrt{s}$, the cross section for the fermionic pair production process $\varphi\varphi \rightarrow \bar{\psi}\psi$ takes the form

$$\sigma(x) = \frac{\pi}{2} \frac{\lambda^2}{x^3} d_H(x) \quad (33)$$

which shows that the mass distribution $d_H(x)$ directly enters into a ‘measurable’ quantity. In Ref. [3, 19, 20] a somewhat related ‘Gedanken experiment’ with a decaying particle was described, in which $d_H(x)$ also emerged as a measurable quantity. Indeed, for a nice example from hadron physics we refer to the radiative decay of the ϕ meson theoretically described with the help of spectral functions in Ref. [21] and experimentally measured in Ref. [22].

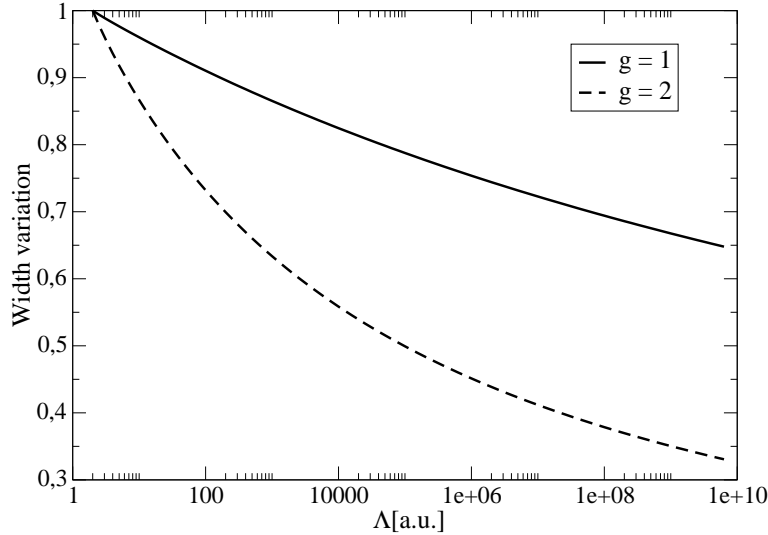


Figure 3: Variation of the Breit-Wigner width as a function of the cutoff and for two values of the coupling. The larger the coupling the faster the width drops for increasing values of the cutoff.

3 Narrowing of the width for increasing cutoff: numerical results

We present now the numerical results for the spectral function of the boson H coupled to fermions via the Lagrangian (3) using the expression (21) with a finite value of the cutoff. For the conceptual purpose followed here, we do not refer to a particular physical system but present the results in terms of an energy unit equal to $2m_f \equiv 1$.

There are five parameters entering in the model: besides the bare parameters m_f , g , $M_{0,H}$, there are also the two wave-function renormalization Z_ψ and Z_H .

In the one-loop study presented here, however, no loop corrections to the fermion field have been evaluated, therefore m_f and Z_ψ do not need any redefinition (m_f is the physical fermionic mass and $Z_\psi = 1$). The quantity Z_H does not need to be redefined, $Z_H = 1$, because for each finite (no matter how large) value of the cutoff Λ the mass distribution of the unstable state H is correctly normalized to unity. (A redefinition of Z_H would be necessary if the cutoff is not kept finite but the limit $\Lambda \rightarrow \infty$ is performed first, see the discussions and the problems in Appendix A.4.) No next-to-leading order for the vertex is considered, therefore g corresponds, in the study here presented, to the physical value of the coupling. Finally, the bare mass $M_{0,H}$ is chosen in such a way that (for a given cutoff Λ) the zero of Eq. (25) takes place at a fixed value of M_H .

We now turn to the spectral function of the unstable boson. We discuss first which is the phenomenological problem we are studying: (i) the fermion mass m_f is known (and the quantity $2m_f$ sets in our model the energy scale) (ii) the cutoff Λ is supposed to be given and to be sizably larger than $2m_f$; (iii) in a fermionic pair production process, the measurement of the cross section, $\sigma(x)$ of Eq. (33), allows to determine the function $d_H(x)$. In particular, one could measure the position of the peak and its height. In this way one can fix the two (remaining) free parameters of the model: g and $M_{0,H}$. Once g and $M_{0,H}$ have been determined in order to reproduce the position and the height of the peak of $d_H(x)$, the function is fixed. In principle, one can compare the rest of its behavior with putative experimental points.

In Fig. 2 we show the spectral function for fixed values of $g = 1$ and $M_H = 1.5$ and for two extreme values of the cutoff: $\Lambda = 2$, close to M_H , and $\Lambda = 2 \times 10^{20}$ a situation reminiscent of

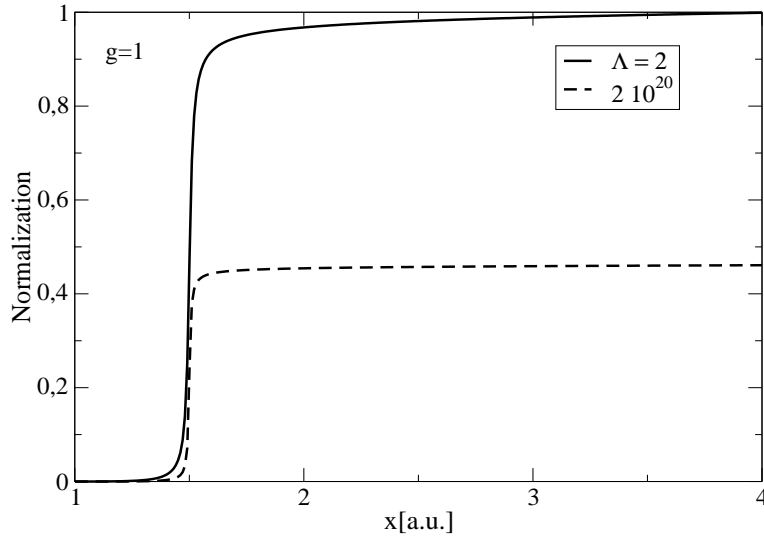


Figure 4: Primitives of $d_H(x)$ for two values of Λ at fixed coupling g .

the standard model of particle physics where the Planck scale is much larger than any mass of the fundamental particles. As explained before, a term in the self energy (21) that mixes the cutoff energy scale and the typical energy scale of the unstable particle is present. Thus, $d_H(x)$ is explicitly dependent on the cutoff Λ : in particular, an increase of the cutoff implies a logarithmic decrease of the width of the spectral function $d_H(x)$. In turn, this means that there is not a full decoupling of the cutoff. Notice also another remarkable property: the height of the peak do not depend on the cutoff, which regulates solely the width of the peak.

To quantify this peculiar behavior we have done the following analysis: we define the Breit-Wigner width W of the particle as the width at half maximum. We calculate W for two values of g as functions of the cutoff. In Fig. 3 we show the corresponding results (the widths have been divided by their values at $\Lambda = 2$ to better appreciate the effect of the coupling on the narrowing). In both cases W decreases as a function of Λ , the larger the coupling the faster is the narrowing of W . This result clearly shows that it is in principle possible to determine the value of an high energy cutoff by ‘measuring’ the spectral function of such an unstable boson. It is also interesting to calculate the primitives of the spectral functions to see how the normalization is “distributed” in the energy range. We show results in Fig. 4: for the small value $\Lambda = 2$, the normalization to one is obviously reached very close to the peak, at $x \sim 3M_H$. On the other hand, for the large value of the cutoff, $\Lambda = 2 \times 10^{20}$, within an energy scale of $\sim 3M_H$ only roughly 50% of the normalization is reached. This is clearly due to the long high energy tail of the spectral function which is obtained in this case. (Note that, as discussed in the Appendix A.4, by removing the cutoff dependent terms, one obtains that most of the normalization is distributed at extremely large energy scales, a situation which is clearly unphysical.)

A side-remark about the time-evolution of the unstable system $|H\rangle$ is in order: the survival probability amplitude $a(t) = \langle H | e^{-iHt} | H \rangle$ is obtained by computing the Fourier transform of the spectral function of the unstable state H [19, 20]. The cutoff, as it has been shown in Ref. [23], regulates the temporal window during which the decay law is not exponential and possible interesting phenomena as the Quantum Zeno and Anti-Zeno effects could arise. That is, in the present example the interval of time, in which the survival probability $p(t) = |a(t)|^2$ deviates from the exponential law, lasts for a time interval of about $1/\Lambda$. Thus, for a very large cutoff, the non-exponential regime elapses for a very short time. This result is different from the corresponding one in the case of a superrenormalizable theory, where the duration of the

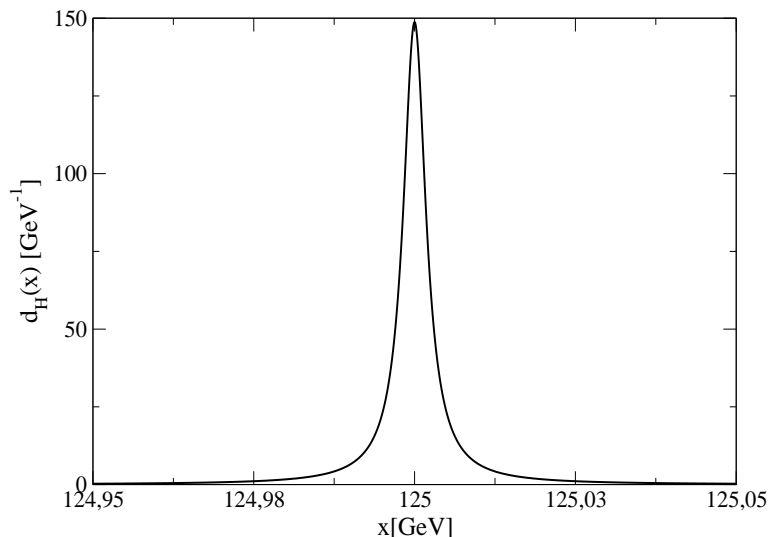


Figure 5: Higgs spectral function obtained by considering only its decay into $b\bar{b}$. No dependence from the cutoff is visible in this case.

non-exponential time interval is sizable and practically independent on the cutoff [19].

As a last result of this Section we study the case of the Higgs boson (see Ref. [24] for a complete treatment of the Higgs boson line shape). The Higgs particle couples to fermions as in Eq. (3): it is therefore interesting to apply our formalism by studying this physical system. For the Higgs mass we use $M_H = 125$ GeV [16]. For this mass, the Higgs decays dominantly in a $b\bar{b}$, then we use $m_f = m_b = 4.18$ GeV [25]. The coupling g can be easily determined as

$$g = \sqrt{3} \frac{m_b}{v}$$

where $v = 246$ GeV is the vacuum expectation value of the Higgs field and $\sqrt{3}$ takes into account the color degree of freedom (not present in our Lagrangian). The tree-level decay width turns out to be $\Gamma_{H \rightarrow \bar{\psi}\psi}^{t-1}(x = M_H) = 4.28$ MeV and is thus much smaller than the Higgs mass. The Breit-Wigner approximations is very good in this case. For these numerical values, when varying the cutoff between the wide range $\Lambda = 10^3$ GeV and 10^{19} GeV, no visible variation of the spectral function is found. The spectral function for these numerical values is presented in Fig. 5. (In order to obtain a visible effect one should consider a very large and unphysical cutoff of about 10^{10000} GeV.)

An hypothetical broad scalar boson, eventually also coupled to the vector bosons, would show more visible effects: in the case of large couplings the influence of the cutoff on the spectral function would be sizable. In some extensions of the Standard model, which go beyond the minimal Higgs coupling, possible other massive scalars and broad particles are predicted, for which the described effects could be seen.

Moreover, many of the properties described above do not depend on the precise form of the Lagrangian in Eq. (3), as long as renormalizability is fulfilled. In fact, the behavior of the decay width $\Gamma \sim x$ would be such in each two-body decay involving a renormalizable interaction. For instance, the coupling of the weak bosons Z^0 and W^\pm to leptons is of such type. In future studies we plan to investigate in more details other renormalizable Lagrangians. On the contrary, the properties change if a superrenormalizable [3, 18] or a non-renormalizable Lagrangian [21] are considered. As already mentioned, in the superrenormalizable case $\Gamma \sim x^{-1}$ and no dependence of the spectral function on a (large) cutoff is visible. On the other hand, for a non-renormalizable theory $\Gamma \sim x^3$ the theory makes sense only if the cutoff is small.

4 Conclusions

In this work we have computed the spectral function of a scalar boson coupled to fermions via resummation of the one loop contributions into the scalar propagator. The propagator satisfies the Källen-Lehman representation and the corresponding spectral function is normalized to unity *when* a finite (no matter how large) cutoff on the energy of the unstable boson is used. The correct normalization is clearly connected with the completeness of the basis of the states into which the unstable particle state is decomposed.

The finite cutoff, in turn, affects the properties of the spectral function: the Breit-Wigner width indeed narrows as the cutoff increases. In a fundamental theory, the existence of a energy cutoff is often connected to a change of degrees of freedom and thus, within the standard model of particle physics, the cutoff would indicate the energy scale at which ‘new Physics’ is expected. Another possibility is the existence of a minimal length such as the one coming from a discrete structure of the space-time. From a phenomenological point of view, the measurement of the line shape of an unstable boson could signal the existence of the cutoff: phenomena occurring at very high energy could influence low energy properties of the system. It turns out that for the Higgs boson, at least the one recently discovered at LHC, the effect of the finite cutoff on the spectral function is too small to be seen. Our results could anyway be important in presence of other Higgs (or other particles beyond the minimal SM) particles having higher masses and widths. Moreover, also within hadronic physics, where the cutoff is related to the scale of confinement, some interesting modification of the spectral function could have a phenomenological relevance, for instance for the medium modification of the ρ meson spectral function which heavy ions experiments are looking for.

There are two possible future directions to be taken: on the theoretical side, one should go beyond the resummed fermionic one-loop approximation considered here. The next-to-leading order correction for the propagator of the unstable state H arises when considering an insertion of a scalar virtual exchange in the fermionic loop. This diagram, which is of order g^4 , should be also resummed. Being the scalar propagator also part of the dressing, one has a problem of the Dyson-Schwinger type, which is obviously more difficult to solve and the renormalization of the charge would be necessary within this context. Considering that the presented considerations about the finiteness of the cutoff are rather general, we do not expect a qualitative change of our results. However, a mathematical proof that this is the case would be very valuable. On the phenomenological side, we plan to compute the spectral functions of vector bosons coupled to fermions, which is clearly relevant for the weak interaction. Namely, the presented phenomena are quite general for each renormalizable theory, and therefore should apply to the weak gauge bosons. In this way we can study the effect of a putative cutoff in the weak sector as well.

In the end, we think that it would be also possible to check our results using lattice Quantum Field Theory. In that case, a cutoff is naturally present (the finite lattice spacing) which resembles closely our finite cutoff used here. The simulation of the simple Yukawa Lagrangian in Eq. (1) should be feasible also in Minkowski space.

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A Details of the calculations

A.1 The decay width

The decay width is explicitly evaluated by making use of the following standard relations:

$$\Lambda_+(\vec{k})_{ab} = \sum_{\alpha} u_a^{(\alpha)}(\vec{k}) \bar{u}_b^{(\alpha)}(\vec{k}) = \left(\frac{\gamma^{\mu} k_{\mu} + m_f}{2m_f} \right)_{ab} , \quad (34)$$

$$\Lambda_-(\vec{k})_{ab} = - \sum_{\alpha} v_a^{(\alpha)}(\vec{k}) \bar{v}_b^{(\alpha)}(\vec{k}) = \left(\frac{-\gamma^{\mu} k_{\mu} + m_f}{2m_f} \right)_{ab} . \quad (35)$$

Out of the latter expressions, one rewrites the squared amplitude as:

$$\sum_{\alpha, \beta} |\mathcal{M}^{\alpha\beta}|^2 = \sum_{\alpha, \beta} g^2 \left[\bar{u}_a^{(\alpha)}(\vec{k}_1) v_a^{(\beta)}(\vec{k}_2) \right] \left[\bar{u}_b^{(\alpha)}(\vec{k}_1) v_b^{(\beta)}(\vec{k}_2) \right]^{\dagger} \quad (36)$$

$$= g^2 \sum_{\alpha, \beta} \left[\bar{u}_a^{(\alpha)}(\vec{k}_1) v_a^{(\beta)}(\vec{k}_2) \right] \left[\bar{v}_b^{(\beta)}(\vec{k}_2) u_b^{(\alpha)}(\vec{k}_1) \right] \quad (37)$$

$$= -g^2 \Lambda_+(\vec{k}_1)_{ba} \Lambda_-(\vec{k}_2)_{ab} = g^2 \text{Tr} \left[\frac{\gamma^{\mu} k_{1\mu} + m_f}{2m_f} \frac{\gamma^{\nu} k_{2\nu} - m_f}{2m_f} \right] \quad (38)$$

$$= g^2 \frac{4(k_1 \cdot k_2) - 4m_f^2}{4m_f^2} = g^2 \frac{8 \left(\frac{x^2}{4} - m_f^2 \right)}{4m_f^2} . \quad (39)$$

where in the last step we have taken into account that

$$k_1 \cdot k_2 = \frac{x^2 - 2m_f^2}{2} . \quad (40)$$

A.2 Regularizations

A.2.1 Cutoff scheme

Here we show which is the formal expression of the nonlocal Lagrangian necessary to generate the cutoff vertex function described in Sec. 2:

$$(\mathcal{L}_{int})_{cutoff} = g \int d^4 z d^4 y_1 d^4 y_2 H(x+z) \bar{\psi}(x+y_1) \psi(x+y_2) \tilde{\phi}(z, y_1, y_2) , \quad (41)$$

where the vertex-function $\tilde{\phi}(z, y_1, y_2)$ in position space has been introduced. The case $\tilde{\phi}(z, y_1, y_2) = \delta(z) \delta(y_1) \delta(y_2)$ delivers the local limit of the Lagrangian (3). (For similar approaches see Refs. [9, 17] and refs. therein.)

By performing the usual steps, we obtain that the vertex function in momentum space is given by the Fourier transform of $\tilde{\phi}(z, y_1, y_2)$:

$$\phi(p, k_1, k_2) = \int d^4 z d^4 y_1 d^4 y_2 e^{ipz} e^{-ik_1 y_1} e^{-ik_2 y_2} \tilde{\phi}(z, y_1, y_2) . \quad (42)$$

Here we assume that $\tilde{\phi}(z, y_1, y_2)$ is such that

$$\phi(p, k_1, k_2) = \phi_{\Lambda} \left(p, q = \frac{k_1 - k_2}{2} \right) = f_{\Lambda} \left(\frac{q^2 p^2 - (q \cdot p)^2}{p^2} \right) . \quad (43)$$

Note, to this end $\tilde{\phi}(z, y_1, y_2)$ must be of the form $\tilde{\phi}(z, y_1, y_2) = \varphi(z, y_1 - y_2)\delta(y_1 + y_2)$. In fact, introducing $y = y_1 - y_2$ and $Y = y_1 + y_2$, one finds

$$\begin{aligned}\phi(p, k_1, k_2) &= \int d^4z d^4y d^4Y e^{ipz} e^{-ik_1y_1} e^{-ik_2y_2} \varphi(z, y) \delta(Y) \\ &= \int d^4z d^4y d^4Y e^{ipz} e^{-i(k_1-k_2)y} e^{-i(k_1+k_2)Y} \varphi(z, y) \delta(Y) \\ &= \int d^4z d^4y e^{ipz} e^{-i2qy} \varphi(z, y) = \phi_\Lambda(p, q) .\end{aligned}\tag{44}$$

Moreover, In Sec. 2 we worked with a bare mass $M_{0,H}$ and a ‘physical’ mass M_H . Alternatively, one could work with the inclusion of counterterms and impose that the quantity M_H entering in the Lagrangian is the nominal mass of the resonance (thus $M_{0,H} = M_H$). In the present case one introduces the counterterm

$$\mathcal{L}_{ct} = -\frac{g^2 \text{Re} \Sigma(M_{0,H})}{2} H^2 .\tag{45}$$

Considering at the one-loop level the Lagrangian

$$\mathcal{L}_{cutoff, counterterms} = \mathcal{L}_0 + (\mathcal{L}_{int})_{cutoff} + \mathcal{L}_{ct} ,\tag{46}$$

where \mathcal{L}_0 describes the free Lagrangian, Eq. (25) takes the modified form

$$x^2 - M_{0,H}^2 + g^2 \text{Re} \Sigma(x^2) - g^2 \text{Re} \Sigma(M_{0,H}) = 0 ,\tag{47}$$

thus implying the solution

$$x = M_H = M_{0,H} .\tag{48}$$

Obviously, nothing substantial would change by following this procedure. Note, the introduction of counterterms can be applied to other regularization schemes as well.

A.2.2 Pauli-Villars

In the Pauli-Villars (PV) approach one subtracts from the original loop of particles with mass m_f a second loop with particles of mass $\Lambda_{PV} \gg m_f$:

$$\Sigma(x) \rightarrow \Sigma_{PV}(x) = \lim_{\Lambda \rightarrow \infty} \left(\Sigma(x) - (\Sigma(x))_{m_f \rightarrow \Lambda_{PV}} \right) .\tag{49}$$

Formally, we can still use the previous expression with the cutoff Λ , but here the condition $m_f \ll \Lambda_{PV} \ll \Lambda$ must hold. In the end each dependence on Λ disappears and its value can be sent to infinity, but the dependence on the new high scale Λ_{PV} is present. The explicit expression for $\Sigma_{PV}(x)$ reads:

$$\begin{aligned}\Sigma_{PV}(x) &= -\frac{(x^2 - 4m_f^2)^{3/2}}{4\pi^2 x} \text{arctanh} \left(\frac{x}{\sqrt{x^2 - 4m_f^2}} \right) + \frac{x^2}{4\pi^2} \ln \left(\frac{\Lambda_{PV}}{m_f} \right) \\ &+ \frac{(x^2 - 4\Lambda_{PV}^2)^{3/2}}{4\pi^2 x} \text{arctanh} \left(\frac{x}{\sqrt{x^2 - 4\Lambda_{PV}^2}} \right) + \text{const} .\end{aligned}\tag{50}$$

As long as $x \ll \Lambda_{PV}$, this result is equivalent to the form (31), also including the term proportional to $x^2 \ln \Lambda_{PV}$. Thus, if one numerically sets $\Lambda = \Lambda_{PV} \gg m_f$ one finds, in the vicinity

of the peak, a behavior which is very similar to the one obtained in the cutoff case: also the narrowing of the spectral function is obtained. However, for values of x comparable to Λ_{PV} , the loop contribution $\Sigma_{PV}(x)$ is modified due to the fact that the additional degree of freedom related to the ‘particle’ with mass Λ_{PV} becomes active.

Moreover, the normalization of the spectral function to unity is not fulfilled. We can easily understand what goes wrong in the present case by writing the modified Lagrangian which delivers the Pauli-Villars formulae:

$$(\mathcal{L}_{H\psi})_{PV} = \frac{1}{2}(\partial_\mu H)^2 - \frac{1}{2}M_{0,H}^2 H^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m_f)\psi + \bar{\psi}_{PV}(i\gamma^\mu \partial_\mu - \Lambda_{PV})\psi_{PV} \\ + gH\bar{\psi}\psi + igH\bar{\psi}_{PV}\psi_{PV} . \quad (51)$$

The new fermion field ψ_{PV} with mass Λ_{PV} is introduced: it should be noticed that, in order to obtain the required cancellation, the coupling of the latter with the boson field H is an imaginary number ig . For this reason, the S matrix is not unitary. As a consequence, the normalization of $d_H(x)$ is lost (the ‘new’ particle gives rise to a negative contribution to the spectral function). In conclusion, the use of the Pauli-Villars scheme delivers similar results to the cutoff scheme as long as Λ_{PV} is finite, but we prefer the latter because it explicitly guarantees the correct normalization of the spectral function to unity. Conversely, sending Λ_{PV} to infinity generates the same problems discussed in Secs. 2.4 and A.4.

A.2.3 Dimensional regularization

Within the dimensional regularization scheme, one calculates the integral of the fermion loop in d dimensions with $d = 4 - \epsilon$ and then takes the limit $\epsilon \rightarrow 0$. In this case the coupling constant g has the dimension of $[\text{Energy}^\epsilon]$, therefore the spectral function should scale as $x^{-1-2\epsilon}$, which is convergent for each -no matter how small- value of ϵ . This is mathematically reminiscent to the finite cutoff case, but the physical interpretation of a nonzero ϵ is not meaningful.

When calculating the self energy $\Sigma(x)$ using the standard formulae (see also Eq. 10.33 of [4]) one finds:

$$\Sigma(x) \rightarrow \Sigma_{DR}(x) = 4i \int_0^1 dy \int \frac{d^d l}{(2\pi)^d} \frac{\Delta + l^2}{(l^2 - \Delta)^2} ; \quad (52)$$

then, after integrating in $d^d l$ we obtain:

$$\Sigma(x)_{DR} = -4 \int_0^1 dy \frac{1}{(4\pi)^{d/2}} \left(\Delta \Gamma(2 - d/2) \frac{1}{\Delta^{2-d/2}} - 2\Gamma(1 - d/2) \frac{1}{\Delta^{1-d/2}} \right) , \quad (53)$$

where $\Delta = m_f^2 - y(1-y)x^2$ and where Γ is the Euler function. Making use of Eqs. A49 and A50 of [4] we obtain:

$$\Sigma(x)_{DR} = -12 \int_0^1 dy \frac{1}{(4\pi)^2} \Delta(2/\epsilon - \gamma + O(\epsilon)) \left(1 - \frac{\epsilon}{2} \log \Delta \right) \quad (54)$$

Calculating the integral and keeping the leading terms in ϵ for the real part we get:

$$\Sigma(x)_{DR} = \frac{x^2 - 6m_f^2}{4\pi^2\epsilon} - \frac{(x^2 - 4m_f^2)^{3/2}}{4\pi^2 x} \text{arctanh} \left(\frac{x}{\sqrt{x^2 - 4m_f^2}} \right) \quad (55)$$

Notice that the imaginary part obtained in this scheme is, as it should, equal to the one obtained in the other schemes when the cutoff is sent to infinity. Comparing the previous equation with Eq. (31) one sees the correspondence $1/\epsilon \propto \ln \Lambda$. The divergence of the real part is as well known linear and non logarithmic and it is usually reabsorbed in the mass and field strength renormalization in the case of stable particles. In the case of unstable particles, in the so called “complex mass renormalization scheme” [26], the renormalization procedure is performed by introducing a complex mass for the resonance. For the Higgs particle for instance, to simplify the calculation, one expands the mass counterterm for $\Gamma_H/M_H \rightarrow 0$, an approximation which definitely holds being the mass of the Higgs only 125 GeV. However, eliminating the term proportional to $1/\epsilon$ is completely equivalent to neglect the term proportional to $x^2 \log \Lambda$ in Eq. (31). It generates many inconsistencies, as mentioned in Sec. 2.4 and shown in Sec. A.4.

A.3 Correct normalization to unity of the spectral function in presence of a cutoff

Let us consider the state $|H\rangle$ as the eigenstate of the unperturbed Hamiltonian H_0 which fulfills the normalization condition $\langle H|H\rangle = 1$. The full set of eigenstates of the Hamiltonian H reads $\{|x\rangle\}$ with $H|x\rangle = x|x\rangle$ and $x \geq 0$. Expressing $|H\rangle$ in terms of $|x\rangle$ implies

$$|H\rangle = \int_0^\infty dx a(x) |x\rangle . \quad (56)$$

The quantity $d_H(x) = |a(x)|^2$ is the ‘spectral function’ which is evaluated in this work as the imaginary part of the propagator (see also Ref. [20] for a more detailed discussion of these relations). It naturally follows that

$$1 = \langle H|H\rangle = \int_0^\infty dx d_H(x) . \quad (57)$$

Taking into account that, in the case of a hard cutoff, $d_H(y)$ vanishes for $y > 2\sqrt{\Lambda^2 + m_f^2}$, the Källen-Lehman representation can be rewritten as

$$\Delta_H(x) = \int_0^{2\sqrt{\Lambda^2 + m_f^2}} dy \frac{d_H(y)}{x^2 - y^2 + i\epsilon} . \quad (58)$$

For $x \gg 2\sqrt{\Lambda^2 + m_f^2}$ no pole is encountered in the integral; at the same time the loop function $\Sigma(x)$ is very small (the real part goes to zero very fast for $x \gg 2\sqrt{\Lambda^2 + m_f^2}$ while the imaginary part is identically zero). It then follows

$$\frac{1}{x^2} = \frac{1}{x^2} \int_0^{2\sqrt{\Lambda^2 + m_f^2}} dy d_H(y) \text{ for } x \gg 2\sqrt{\Lambda^2 + m_f^2} , \quad (59)$$

that is:

$$\int_0^{2\sqrt{\Lambda^2 + m_f^2}} dy d_H(y) = \int_0^\infty dy d_H(y) = 1 . \quad (60)$$

We now turn to the case of a smooth cutoff function, which assures that the loop contribution $\Sigma(x)$ is very small beyond a certain energy scale Λ . However, the imaginary part, and so the

spectral function, do not vanish exactly for $x \gg \Lambda$. The proof of the correct normalization is in this case more difficult. As a first step, we decompose the integral as

$$\Delta_H(x) = \int_0^{m_f\sqrt{x}} dy \frac{d_H(y)}{x^2 - y^2 + i\varepsilon} + \int_{m_f\sqrt{x}}^\infty dy \frac{d_H(y)}{x^2 - y^2 + i\varepsilon}. \quad (61)$$

When $m_f\sqrt{x} \gg \Lambda$ (which implies also $x \gg \Lambda$) the propagator is $\Delta_H(x) = 1/x^2$. We thus obtain:

$$\frac{1}{x^2} = \frac{1}{x^2} \int_0^{m_f\sqrt{x}} dy d_H(y) + P \int_{m_f\sqrt{x}}^\infty dy \frac{d_H(y)}{x^2 - y^2 + i\varepsilon} \quad (62)$$

where P stands for principal part. In the large energy limit the spectral function $d_H(y)$ can be approximated as

$$d_H(y) \simeq \frac{g^2}{8\pi y} f_\Lambda^2 \left(-\sqrt{\frac{y^2}{4} - m_f^2} \right) \simeq \frac{g^2}{8\pi y} f_\Lambda^2 (-y/2). \quad (63)$$

Finally, taking the limit $x \rightarrow \infty$ we obtain

$$1 = \int_0^\infty dy d_H(y) + \lim_{x \rightarrow \infty} \left[x^2 P \int_{m_f\sqrt{x}}^\infty dy \frac{g^2}{8\pi y} \frac{f_\Lambda^2(-y/2)}{x^2 - y^2 + i\varepsilon} \right]. \quad (64)$$

If the cutoff function is such that

$$\lim_{x \rightarrow \infty} \left[x^2 P \int_{m_f\sqrt{x}}^\infty dy \frac{g^2}{8\pi y} \frac{f_\Lambda^2(-y/2)}{x^2 - y^2 + i\varepsilon} \right] = 0 \quad (65)$$

it follows that the correct normalization condition holds:

$$1 = \int_0^\infty dy d_H(y). \quad (66)$$

Indeed, as long as $f_\Lambda^2(-y/2)$ falls off sufficiently fast, Eq. (65) is fulfilled. A power-like or exponential decrease introduced to assure the convergence of the loop integral automatically implies the validity of Eq. (65), hence the correct normalization to unity of the spectral function (independently, also here, on the precise value of the cutoff).

A.4 Removing completely the Λ dependence

We describe here the standard renormalization of the Lagrangian under study in the case of unstable particles. The starting point is the loop expression in Eq. (31). First, we rewrite the real and the imaginary part of $\tilde{\Sigma}(x)$ as follows:

$$\text{Re}[\tilde{\Sigma}(x)] = A(x) + \frac{\Lambda^2}{2\pi^2} - \frac{6m_f^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) + \frac{x^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) \quad (67)$$

$$\left| \text{Im}[\tilde{\Sigma}(x)] \right| = \frac{x}{g^2} \Gamma_{H \rightarrow \bar{\psi}\psi}^{t-1}(x) = \frac{\left(\frac{x^2}{4} - m_f^2 \right)^{3/2}}{\pi x} \theta(x - 2m_f). \quad (68)$$

where the cutoff independent quantity $A(x)$ is given by

$$A(x) = -\frac{(x^2 - 4m_f^2)^{3/2}}{8\pi^2 x} \ln \left(\frac{x + \sqrt{x^2 - 4m_f^2}}{x - \sqrt{x^2 - 4m_f^2}} \right). \quad (69)$$

The propagator takes the form

$$\tilde{\Delta}_H(x) = \left[x^2 - M_{0,H}^2 + g^2 A(x) + \frac{g^2 \Lambda^2}{2\pi^2} - \frac{6g^2 m_f^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) + g^2 \frac{x^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) + ig^2 \text{Im}[\tilde{\Sigma}(x)] + i\varepsilon \right]^{-1}. \quad (70)$$

The renormalized mass $M_{H,ren}$ is defined as the solution of the equation in a similar way as Eq. (25):

$$M_{H,ren}^2 - M_{0,H}^2 + g^2 A(M_{H,ren}) + \frac{g^2 \Lambda^2}{2\pi^2} - \frac{6g^2 m_f^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) + g^2 \frac{M_{H,ren}^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) = 0. \quad (71)$$

By performing a Taylor expansion of the real part around $M_{H,ren}$ we obtain

$$\tilde{\Delta}_H(x) = \left[(x^2 - M_{H,ren}^2) \left(1 + g^2 (\partial_{x^2} A)_{x^2=M_{H,ren}^2} + \frac{g^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) \right) + g^2 \tilde{A}(x) + ig^2 \text{Im}[\tilde{\Sigma}(x)] + i\varepsilon \right]^{-1} \quad (72)$$

where

$$\tilde{A}(x) = A(x) - A(x = M_{H,ren}) - (\partial_{x^2} A)_{x^2=M_{H,ren}^2} (x^2 - M_{H,ren}^2). \quad (73)$$

By introducing the wave function renormalization $H \rightarrow \sqrt{Z_H} H$ the propagator takes the form

$$\tilde{\Delta}_H(x) = \frac{1}{Z_H} \left[(x^2 - M_{H,ren}^2) \left(1 + g^2 (\partial_{x^2} A)_{x^2=M_{H,ren}^2} + \frac{g^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right) \right) + g^2 \tilde{A}(x) + ig^2 \text{Im}[\tilde{\Sigma}(x)] + i\varepsilon \right]^{-1} \quad (74)$$

$$= \frac{1}{Z_H K} \left[(x^2 - M_{H,ren}^2) + \frac{g^2}{K} \tilde{A}(x) + i \frac{g^2}{K} \text{Im}[\tilde{\Sigma}(x)] + i\varepsilon \right]^{-1}. \quad (75)$$

where the (formally divergent) quantity K reads

$$K = 1 + g^2 (\partial_{x^2} A)_{x^2=M_{H,ren}^2} + \frac{g^2}{4\pi^2} \ln \left(\frac{2\Lambda}{m_f} \right). \quad (76)$$

Now, one can need also to perform a renormalization of the coupling $g \rightarrow g_{ren}$ (which is obtained here through multiplicative constant, thus no running coupling arises at this level), leading to the following equations for Z_H and g_{ren} :

$$Z_H K = N, \quad (77)$$

$$\frac{g^2}{K} = g_{ren}^2 \quad (78)$$

whereas N and g_{ren} are finite constant. In this way the propagator takes the renormalized form

$$\tilde{\Delta}_H(x) = \frac{1}{N} \left[(x^2 - M_{H,ren}^2) + g_{ren}^2 \tilde{A}(x) + ig_{ren}^2 \text{Im}[\tilde{\Sigma}(x)] + i\varepsilon \right]^{-1} \quad (79)$$

in which each dependence on Λ has been eliminated. One might think that each problem is solved here, but this is not the case. To show it we turn our attention to the spectral function:

$$\tilde{d}_H(x) = \lim_{\varepsilon \rightarrow 0^+} \frac{2x}{\pi} \left| \text{Im } \tilde{\Delta}_H(x) \right|. \quad (80)$$

For $x \gg 2m_f$ (that is, away from threshold) the following simplifications are valid:

$$\tilde{A}(x) = -\frac{x^2}{4\pi^2} \ln x, \quad (81)$$

$$x\Gamma_{H \rightarrow \bar{\psi}\psi}^{\text{t-l,ren}}(x) = \frac{x^2}{8\pi}. \quad (82)$$

Then, the spectral function for $x \gg 2m_f$ is approximated by the following expression:

$$\tilde{d}_H(x) \simeq \frac{1}{N} \frac{g_{ren}^2 x^3}{8\pi} \frac{1}{\left(x^2 - \frac{g_{ren}^2 x^2}{4\pi^2} \ln x\right)^2 + \left(\frac{x^2 g_{ren}^2}{8\pi}\right)^2}. \quad (83)$$

Let us define the point x_* as the solution of the following transcendental equation:

$$1 = \frac{g_{ren}^2}{4\pi^2} \ln x_* \rightarrow x_* = e^{\frac{4\pi^2}{g_{ren}^2}}. \quad (84)$$

Then, for $2m_f \ll x \lesssim x_*$ the function $\tilde{d}_H(x)$ scales as $1/x$ and for $x \gtrsim x_*$ the logarithm starts to dominate and $\tilde{d}_H(x)$ scales as $1/(x \ln^2 x)$, which assures a (slow) convergence of the integral $\int_0^\infty d_H(x) dx$. A numerical evaluation shows that the following approximately scaling law holds:

$$\int_0^\infty \tilde{d}_H(x) dx \simeq \frac{80}{N g_{ren}^2}. \quad (85)$$

At a general level we can immediately discuss two basic problems of the spectral function $\tilde{d}_H(x)$:

(a) The “distribution” of the normalization in the energy range of the particle is spread over very large value of x . In Fig. 6, we show the quantity $F(x) = N \int_{100}^x \tilde{d}_H(y) dy$ for two values of the coupling constant g (the value 100 as the lower limit of integration corresponds to a value much larger than the peak position and is suited to study the energy localization of the state faraway from the peak). Notice first that the saturation of $F(x)$ is reached only at extremely high energy, very far from the nominal mass of the particle. This fact is clearly connected to the slow logarithmic convergence of the integral of the spectral function.

(b) The effect of changing coupling is evident: the smaller the coupling g_{ren} , the larger the normalization is and the later is reached. This property also implies that the small g_{ren} limit is completely at odds with the basic expectation of having the unstable particle mostly localized around the peak.

In order to evaluate the spectral function one has to determine the normalization condition N . To this end, it is useful to recall the case of a stable scalar state: $M_{H,ren} < 2m_f$, which implies $\text{Im}[\tilde{\Sigma}(x = M_{H,ren})] = 0$. In this case, the requirement is that the free propagator with residue 1 at the pole is obtained, thus: $N = 1$, which is a clear and physically meaningful requirement: The state H is a stable asymptotic state entering, for instance, as initial or final state in a two-body process, for which the canonical normalization holds. However, in our case we deal with an unstable state for which $M_{H,ren} > 2m_f$: there is no pole below threshold and it is not clear which condition should be used. Two possibilities are the following:

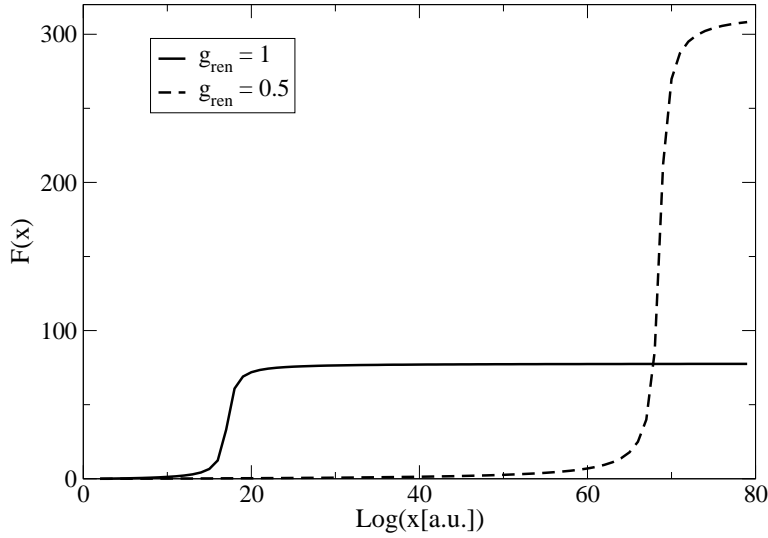


Figure 6: The integral function $F(x)$ is displayed for two values of the coupling. The normalization of the spectral function (not normalized to one) is obtained at extremely large energies.

(i) $N = 1$. In this way the coefficient multiplying the term $(x^2 - M_{H,ren}^2)$ in the denominator of the propagator is unity; this represents a simple generalization of the stable case. However, setting $N = 1$ means that the normalization to unity of $\int_0^\infty \tilde{d}_H(x)dx$ is in general lost and one violates a very basic property of Quantum Mechanics. We regard this ‘solution’ as unphysical. It would rather corresponds to an *ad hoc* prescription to ignore the problems. Notice that within this prescription the curve in the vicinity of the peak looks very similar to the case of a not too large but finite cutoff.

(ii) One can set N as being dependent on g , in such a way that $\int_0^\infty \tilde{d}_H(x)dx = 1$ is fulfilled. Still, as discussed above, the amount of the integral in the vicinity of the peak represents only a very small contribution to the normalization of the spectral function (due to the slow convergence of the latter). Additionally, if we aim to describe the situation in which the spectral function has a certain given (putative measured) height for $x = M_{H,ren} = 1.5$, one runs into problems because the quantity $\tilde{d}_H(x = M_{H,ren})$ is practically independent on g_{ren} which is a quite unrealistic feature.

In conclusion, we believe that the here outlined procedure is not physical.

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